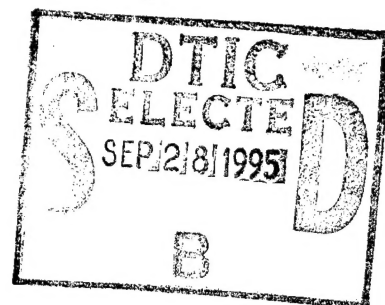# Semi-Annual Report
ONR Grant Number N00014-92-J-1706
Sensitivity of Wavelet Algorithms in Signal Processing

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During the second year of funding under this grant substantial progress has been made in the area of optical flow computations. Following a detailed survey of optical methods for a specially designed set of target tracking problems, a new iterative refinement procedure was developed that compensates for errors in the estimated flow velocity induced by finite difference approximations of spatial and temporal image derivatives. When coupled with a total least squares (TLS) solution scheme (as opposed to a more standard least squares method) and an aggregate velocity assignment for distant targets, this approach yields very accurate flow estimates at a fraction of the computational cost of competing methods. This approach has the pleasant feature that it admits a simple convergence analysis showing that temporal and spatial smoothing improves the iterative convergence for the special case of convective flow. This work has recently been supplemented with a "size band limited" image processing filter based on partial differential operators; this approach has the advantage that it is considerably simpler than a similar scheme based on morphilogical set filters.

The publications list is followed by a short narrative section that describes the research that has been done so far under this grant and our plans for the coming year.

[1] Hewer, G., C. Kenney, W. Kuo, "Optical Flow Methods for Tracking Problems," *Proceedings SPIE Conference*, April 1994, Orlando, Florida, Vol. 2242, pp. 561-572.

[2] Hewer, G., C. Kenney, W. Kuo, "Wavelets, Curvature and Chaining Issues With Applications to the Computation of Optical Flow," *Proceedings SPIE Conference*, July 1994, San Diego, California

[3] Kenney, C., and A.J. Laub, "Small-Sample Statistical Condition Estimates for General Matrix Functions," *SIAM J. Sci. Comput.*, 15(1994), pp. 36-61.

[4] Gudmundsson, T., C. Kenney, and A.J. Laub, "Small-Sample Statistical Estimates for Matrix Norms," to appear in *SIAM J. Matr. Anal. Appl.*, 1995.

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[5] Kenney, C.S., and A.J. Laub, "A Statistical Approach to Condition Estimation," *Proc. 32nd IEEE Conf. on Decision and Control*, San Antonio, Texas; December 1993; pp. 3156–3161.

[6] Kenney, C.S. and G.A. Hewer, "Simple Algorithms for Affine Invariant Image Processing," *IEEE Trans. Pattern Analysis and Machine Intelligence*, submitted.

[7] Hewer, G., C. Kenney, W. Kuo, "Size-Limited Band Filters Using Affine Invariant Tophat Operators," in preparation.

[8] Kenney, C.S. and P.L. Overfelt, "A Simple Method of Generating a Basis of Solutions for Linear Partial Differential Equations," *Mathematics Notes* (Dr. Carl Baum, Editor), Note 90, Phillips Laboratory, Kirtland AFB, 1993.

[9] Holthaus, M., C.S. Kenney, and A.J. Laub, "Numerical Methods for Studying Parameter Dependence of Solutions to Schrödinger's Equation," in *Differential Equations, Dynamical Systems, and Control Science* (A Festschrift in Honor of Lawrence Markus), (K.D. Elworthy, W.N. Everitt, and E.B. Lee, eds.), Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, June 1993.

[10] Hench, J.J., C.S. Kenney, and A.J. Laub, "Methods for the Numerical Integration of Hamiltonian Systems," *Circuits, Systems, and Signal Processing*, 13(1994), pp.695-732.

[11] Kenney, C.S. and G.A. Hewer, "Trace Norm Bounds for the Stable Inverse Lyapunov Operator," *Lin. Alg. Appl.*, to appear.

[12] Kenney, C., and A.J. Laub, "A Hyperbolic Tangent Identity and the Geometry of Padé Sign Function Iterations," *Numerical Algorithms*, 7(1994), pp. 111-128.

[13] Kenney, C., and A.J. Laub, "The Matrix Sign Function," *IEEE Trans. Automat. Contrl.*, submitted.

[14] Hewer, G., B.Lucas, C. Kenney, J. Martin, and P. Overfelt, "Scaling Function Identification and Approximation," *IEEE Trans. Pattern Analysis and Machine Intelligence*, submitted.

Narrative

Brief descriptions of the research performed under ONR Grant N00014-92-J-1706 are given in this section with citations keyed to the above list. A more detailed exposition can be found in the original grant request.

1) Optical Flow Computation: In [1] the results are given of a long term project aimed at uniform evaluation of optical flow algorithms on a set of problems related to target tracking for missiles. The problem set contains a variety of parameter dependent problems involved in acquisition and tracking after acquisition especially in the presence of cluttered environments. The set of basic algorithms tested includes normal flow subject to smoothness constraints in the manner of Horn and Schunck, edge dectection via zero crossings as per Duncan and Chou, estimating multiple component motion as per Bergen et al. and window matching across frames using differencing, correlation or more computational intensive distance measures such as the Hausdorff distance between sets.

Each of the algorithms tested was been coupled with a multi-resolution scheme in which the individual sequence images are subjected to a scale space type image processing. The optical flow is determined at each scale with subsequent comparison of flow results across scales. Multi-resolution schemes generally fall into two major classes consisting of wavelet decompositions and scale space decomposition based on the evolution of the image under a partial differential operator. A nice example of the latter class is the commonly used scale space resulting from Gaussian smoothing with the smoothing variance corresponding to the scale variable. This in turn is equivalent to image evolution under the heat operator. More recently an highly nonlinear partial differential operator has been used in affine invariant image processing. Related operators which depend on the curvature of the image isoclines have been included in the scale space testing.

Details on the optical flow algorithms and multiscale resolutions are described in [1], along with numerical results and plans for future research in this area.

Smoothness constraints for optical flow often involve computations over a 2-D region, which may be the entire image frame. Hildreth has proposed an optical flow computation based on using the edges in an image sequence. In essence this is a 1-D version of Horn and Schunck's algorithm. Sundareswaran and Mallat combine Hildreth's approach with the multiscale information to regularize the optical flow computation. Edge based optical flow methods have several advantages: (1) they have the potential to overcome the aperture problem that results from the underdetermined nature of the constant brightness assumption, and (2) they greatly reduce the computational burden over methods that rely on global smoothing constraints since these usually lead to linear systems on the order of the number of pixels in the entire image.

In [2] we discuss Hildreth's approach and the Sundareswaran and Mallat's algorithm as well as a new approach based on aggregate velocity and iterative refinement. In this approach we assume that the object of interest has edge points with nearly equal optical flow vectors. This is often the case for example with distant objects in a targeting environment, at least in the initial stages of tracking. The aggregate velocity assumption yields an overdetermined system of equations with two unknowns (the x and y components of the group velocity). The matrix of coefficients and the data

vector for this overdetermined system may contain errors as a result of approximating temporal and spatial derivatives. Because of this it is appropriate to solve this system via the total least squares approach rather than the more common least squares method. This is described in more detail in [2]. Because there are only two unknowns, the computation of the aggregate velocity vector is very fast. The computed velocity may still be in error because of the derivative approximation errors. Because of this we introduce an iterative correction procedure that subtracts the computed velocity from the image sequence and then recomputes a new aggregate velocity that in turn may be subtracted from the new image sequence. This is repeated until the velocity stabilizes.

An important question is whether indeed this method does converge. This problem admits analysis for the case of one dimensional convected flow and conditions ensuring convergence are easily formulated (see [2]). Moreover we show that smoothing (either temporal or spatial) enhances convergence without changing the flow velocity. Numerical examples are presented illustrating the main points of our procedure for both one and two dimensional problems.

2) Sensitivity and Condition Estimation: One of the main goals of this grant has been to develop a general procedure for measuring the sensitivity of wavelet and signal processing algorithms. Such a procedure should be able to provide an efficient assessment of the reliability of computed values from these algorithms. Fortunately, the work in [3] forms a basis for this type of estimation procedure for multivariable scalar valued functions $f : R^n \rightarrow R$. For these functions, the norm of the gradient provides a measure of the sensitivity of the function at the point of evaluation. The main problem then is to get an estimate of the norm of the gradient without having to evaluate the gradient itself, since this would require n function evaluations if we use finite differences to approximate the n partial derivatives in the gradient. The saving grace of this problem is that, by using the theory of random inner products on the unit sphere in n dimensions, we can estimate the norm of the gradient with only a few function evaluations (typically two or three) and that exact probabilistic bounds can be given on the accuracy of the estimate. For example, condition estimates are considered acceptable if they are within a factor of 10 (because we are interested in estimating the number of digits of accuracy). If three function evaluations are used in estimating the condition number of a scalar valued function, then the probability that the estimate is within a factor of 10 is 0.9989 and k function evaluations lead to k^{th} order condition estimates, i.e. the probability of being off by a factor ω is less than a constant times $1/\omega^k$. A very readable account of this work has appeared in [4].

This "small-sample" statistical condition estimation method has been extended to matrix valued functions by the work in [5]. Here there are two possible approaches: 1) treat the individual entries of the function as if they were scalar valued functions and apply the scalar condition estimates, or 2) estimate the overall condition of the

functional map via the norm of the Fréchet derivative. The first approach has the nice feature that the scalar theory can be applied immediately and that estimates of the conditioning of all the entries can be obtained at the same time with only a few function evaluations. However, if the number of outputs is large then it is often more convenient to provide just one number as an overall condition measure, as in the second approach. This overall assessment has been considered in [5] in which a conjecture is presented relating the theory of the accuracy of the individual estimates with that of the overall condition estimate. A conservative form of this conjecture is proved in [5], but the accuracy estimates of this conservative appraoch are too pessimistic in the case of a very large number of inputs. Although extensive numerical tests show that it applies in many example problems, more work is needed in this area to establish the full conjecture.

3) Wavelets and Differential Equations: Wavelets and differential equations have natural connections from two points of view: 1) many physical processes have self similar properties under translation and rescaling, and 2) recent work (L. Alvarez, F. Guichard, P. Lions and J. Morel, "Axiomatic and Fundamental Equations of Image Processing," preprint) shows that the image processing schemes which are invariant with respect to certain image operations, such as transposition or rotation, can be viewed as time slice projections of differential operators. As an example, the heat operator gives rise to smoothing or blurring image processors that are equivalent to Gaussian filtering. Some care must be taken in the numerical solution of affine invariant nonlinear differential equations; the work in [6] shows that standard finite difference implementations can produce erroneous results near extrema. The family of exact solutions in [6] points the way to avoiding these problems and at the same time leads to a nonlinear version of the Courant stability relation between the spatial grid size and the scaling step size.

The affine partial differential operator appears to be particularly useful in filtering that is band limited with regard to object size using tophat operators based on PDEs. In this procedure, evolution of the image under the PDE operator is identified with morphological erosion and backwards time evolution is identified with dilation. The composition of these operations is subtracted from the original image to give the tophat operator. See [7] for a discussion of tophat operators and morphological filters.

More specifically, let $u = u(x, y, 0)$ be an image and define the scale space resolution of u as the solution $u = u(x, y, s)$ to the initial value PDE

$$\frac{du}{dt} = Lu, \tag{1}$$

with $u(x, y)$ at $s = 0$ equal to $u(x, y, 0)$. Here L is a spatial partial differential operator. For the affine invariant operator

$$L_{\text{affine}} = (u_{xx}u_y^2 + u_{yy}u_x^2 - 2u_{xy}u_xu_y)^{1/3}. \tag{2}$$

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Let $F_s(w)$ denote an image w after it has evolved for a time period s under the PDE. The tophat operator is defined by

$$T_s(u) = u - F_{-s}(F_s(u)). \qquad (3)$$

That is first we let u evolve for time s, then we reverse the evolution and subtract this result from the original image.

To provide some motivation, we note that under the affine invariant tophat, a peak of radius r with circular symmetry will disappear at $s = \frac{3}{4}r^{4/3}$ since this is how long it takes the circular level curve of radius r to move inward to the center of the peak. Thus any isolated peak in the original image that is of radius less than or equal to r will disappear by the time $s = \frac{3}{4}r^{4/3}$. When the affine PDE is then reversed and run backward small peaks (of radius $\leq r$) are not recreated because the information necessary to do the restoration has disappeared. Thus these peaks are missing from $F_{-s}(F_s(u))$, while peaks of larger radii are restored. Note: only the outer rings of these larger peaks are restored; the inner parts of these peaks are destroyed just as the smaller peaks are. In particular the tophat operator uses this selective destruction to eliminate peaks beyond a given radius; by differencing tophat operators of different radii we obtain size limited band filters.

The connection between image processing filters and evolution under a partial differential operator opens up the possibility of studying the sensitivity of wavelet and signal processing algorithms via the sensitivity of solutions to differential equations and vice versa. The general question of the sensitivity of solutions to differential equations has also taken on increased importance to researchers at the NAWC because of the problem of obtaining accurate computer predictions of variables related to radome electromagnetic problems, such as the boresight error problem. We are especially interested in extending the sensitivity analysis of [3]-[5] to the methods developed in [8]-[10] for solving a variety of problems in differential equations. Likewise the work in [11] opens up new avenues for sensitivity estimation in differential problems related to control theory.

4) Research Directions for 1994-1995: Optical Flow Methods Optical flow algorithms for detecting motion in cluttered environments have assumed new importance in missile tracking problems at the Naval Air Warfare Center in China Lake, especially in the area of thermal imaging against a hot background as opposed to 'blue-sky' tracking. The essential problem to be treated is the reduction in the amount of computation needed to obtain smooth flow fields via global constraints on the allowed variation in the field. Work on this problem has shown that the iterative smoothing procedures needed to solve the global coupling problems can be greately speeded up by working in a cascade of resolution levels such as those provided naturally by wavelet decompositions.

A second approach to the problem of computational reduction involves translating the two-dimensional imaging problem into a one-dimensional problem by focusing only on image aspects such as edges and corners. Thus the identification of such

structures for example by tophat operators takes on new urgency. An important problem in this area is the invariance of the computed optical flow field with respect to reflections in the x and y axes; invariance of this type seems to depend on the choice of the wavelet filters that are used in the processing especially with regard to their symmetry with respect to reflections in x and y. Research in this area will focus on eliminating invariance problems from existing edge detection optical flow algorithms and enhancing their resolution by modifying the wavelet filters to correspond to the image, i.e. using the techniques of selecting the best wavelet to improve the optical flow calculations.

Research in this area will be conducted in conjunction with an ongoing effort at NAWC under the direction of Dr. Gary Hewer.

Abstracts

The following pages give abstracts from papers accepted or submitted under ONR Grant Number N00014-92-J-1706.

A Survey of Optical Flow Methods for Tracking Problems

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ABSTRACT

Results are presented of a numerical survey of optical flow algorithms for tracking problems associated with infrared imaging in high speed missles. The algorithms tested include those associated with normal flow subject to global smoothness constraints, edge detection via zero crossings of the image after convolution with spatio-temporal filters, and windowed matching techniques. The tracking problems considered in this survey fall into two classes: acquisition and tracking after acquisition. These classes can be further divided into near and far range problems, characterized by extended and point target images. Other parameters of interest model allowable target and sensor motion and the amount of background clutter. Each of the above methods for determining optical flow can be used in conjunction with a variety of image preprocessing techniques such as kernel smoothing (especially by Gaussian kernels), and evolution under affine invariant partial differential operators. These preprocessing methods can also be used in combination with other approaches such as temporal layering in which succesive images are combined to produce images with streaks whose edges are predominantly parallel to the optical flow.

2. INTRODUCTION

In this paper we report on a long term project aimed at uniform evaluation of optical flow algorithms on a set of problems related to target tracking for missiles. The problem set contains a variety of parameter dependent problems involved in acquisition and tracking after acquisition especially in the presence of cluttered environments. The set of basic algorithms tested so far includes normal flow subject to smoothness constraints in the manner of Horn and Shunk [15], edge dectection via zero crossings as per Duncan and Chou [9], estimating multiple component motion as per Bergen et al. [6], and window matching across frames using differencing, correlation or more computational intensive distance measures such as the Hausdorff distance between sets [16]. The set of optical flow algorithms will be expanded in the future to include other approaches such the generalizations by Schnörr [30] of the smoothness functionals of Horn and Schunck [15] and Nagel and Enkelmann [23], Kalman filtering time-to-collision estimation of [22] and other Kalman procedures [31], edge detection based on phase information [11] as well as affine invariant movie analysis methods [12].

Each of the algorithms tested has been coupled with a multi-resolution scheme in which the individual sequence images are subjected to a scale space type image processing. The optical flow is determined at each scale with subsequent comparison of flow results across scales. Multi-resolution schemes generally fall into two major classes consisting of wavelet decompositions and scale space decomposition based on the evolution of the

Wavelets, Curvature and Chaining Issues with Applications to the Computation of Optical Flow

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ABSTRACT

Optical flow is an estimate of the velocity field based on the change of intensity patterns in successive images, and is an important quantity in computational vision for dense images. Because of the aperture problem optical flow computations can be ill-posed. This problem is compounded by derivative estimation errors. This paper presents an aggregate velocity scheme that uses iterative velocity refinement along object edge contours obtained via the Mallat-Zhong-Hwang wavelet and chaining algorithms. By working with edge information and aggregate velocities we avoid the aperture problem; iterative refinement compensates for errors in the derivative estimation. Our approach assigns a common velocity to the edge points of an image. When combined with a constant brightness assumption this yields an overdetermined set of linear equations. Since the data vector and matrix coefficients of this linear system consist of temporal and spatial derivative estimates, respectively, and both are subject to errors, the overdetermined system is solved using a total least squares approach. The resulting velocity estimate is then subtracted from the image sequence and the velocity estimation procedure is repeated for the new image sequence. This approach is very fast and accurate for images that have nearly the same edge velocity vectors as is usually the case for distant objects. A convergence analysis is given for the special case of one dimensional convected flow and it is shown that spatial and/or temporal smoothing enhances the convergence.

I. INTRODUCTION

Optical flow is an estimate of the velocity field based on the change of intensity patterns in successive images, and is an important quantity in computational vision in dense images. A well-known method for optical flow computation is the optimization approach introduced by Horn and Schunck [4]. In this approach, the sum of two cost functionals of the unknown flow vectors is minimized. One of the cost functional is the "brightness constraint". The

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[0]Approved for public release; distribution is unlimited.

Small-Sample Monte Carlo Condition Estimates
for General Matrix Functions *

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Abstract

A new condition estimation procedure for general matrix functions is presented that accurately gauges sensitivity by measuring the effect of random perturbations at the point of evaluation. The number of extra function evaluations used to evaluate the condition estimate determines the order of the estimate. That is, the probability that the estimate is off by a given factor is inversely proportional to the factor raised to the order of the method. Comparison is given with the power method of condition estimation which because of its sequential nature can potentially require twice the computational effort, as is the case for the problem of estimating the sensitivity of the matrix exponential. A group of examples illustrates the flexibility of the new estimation procedure in handling a variety of problems and types of sensitivity estimates, such as mixed and componentwise condition estimates.

Keywords: Monte Carlo, conditioning, matrix functions
AMS(MOS) subject classification: 65F35, 65F30, 15A12
Abbreviated title: Monte Carlo Condition Estimates

1 Introduction

If a scalar real-valued function f has a large derivative at $x \in \mathbb{R}$ then the relationship

$$f(x + \delta) = f(x) + f'(x)\delta + O(\delta^2)$$

shows that small perturbations in x can lead to relatively large changes in f. The idea that the magnitude of the derivative provides a measure of the sensitivity or conditioning of f carries over to higher-dimensional mappings [5], [21], [31], [29], [32], [28], [35], [40]. For example, if f maps $\mathbb{R}^n$ into $\mathbb{R}$, the gradient of f at x is the row vector $\nabla f(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n)$ and the Taylor expansion of f has the form

$$f(x + \delta z) = f(x) + \delta \nabla f(x) z + O(\delta^2) . \tag{1}$$

Small-Sample Statistical Estimates for Matrix Norms[*]

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Abstract

This paper extends a recent statistically based vector-norm estimator to matrices. The new estimator requires only a few matrix-vector multiplications and can be applied when the matrix is not known explicitly. It is useful for efficiently estimating the sensitivity of vector-valued functions and can be applied to many problems where the power method runs into difficulties. Lower bounds for the probability that an estimate is within a given factor of the correct norm are derived. These bounds are straightforward to compute and show that a very inaccurate estimate is extremely unlikely in most cases. A conservative lower bound has been derived and a tighter bound is given in the form of a conjecture. This conjecture is true in some important special cases and the general case is supported by considerable empirical evidence.

Keywords: Conditioning, matrix functions
AMS(MOS) subject classification: 65F35, 65F30, 15A12
Abbreviated title: Statistical Condition Estimates

1   Introduction

A novel method for efficiently estimating the sensitivity of a scalar function at a point was recently introduced in [17]. This method is based on implicitly projecting an approximate gradient of the function onto a uniformly randomly chosen low-order subspace, and then computing the norm of the result. Properly scaled, this gives an estimate of the norm of the gradient, and thereby the condition number of the function at the estimation point. The method requires only the evaluation of the function at this point and a few nearby points.

This paper extends the results of [17] to the estimation of the Frobenius norm of the Jacobian of a general vector-valued function. Thus, let $f : \mathbb{R}^n \to \mathbb{R}^t$ be a function differentiable at a point z, and define the Jacobian at z as

$$J(z) = \frac{\partial f}{\partial z}(z).$$

A Statistical Approach to Condition Estimation *

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Abstract

A new approach is presented for estimating the conditioning of general matrix functions by measuring the effect of random perturbations at the point of evaluation. This method is efficient in the sense that the number of extra function evaluations used to evaluate the condition estimate determines the order of the estimate. That is, the probability that the estimate is off by a given factor is inversely proportional to the factor raised to the order of the method. The "transpose-free" nature of this new method allows it to be applied to a much broader range of problems than the commonly used power method of condition estimation. A group of examples illustrates the flexibility of the new estimation procedure in handling a variety of problems and types of sensitivity estimates, such as mixed and componentwise condition estimates. Short MATLAB routines are included to demonstrate the ease with which the new condition method can be implemented in a general setting.

1. Introduction

If a scalar real-valued function f has a large derivative at $x \in \mathbb{R}$, then small perturbations in x lead to relatively large changes in f. The idea that the magnitude of the derivative provides a measure of the sensitivity or conditioning of f carries over to higher-dimensional maps [3], [14], [22], [20], [23], [19], [28], [31]. For example, if f maps $\mathbb{R}^n$ into $\mathbb{R}$, the gradient of f at $x \in \mathbb{R}^n$ is the row vector $\nabla f(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n)$, and

$$f(x + \delta z) = f(x) + \delta \nabla f(x) z + O(\delta^2), \tag{1}$$

where $\|z\| = 1$ with $\|\cdot\|$ denoting the vector 2-norm defined by $\|z\|^2 = \Sigma |z_i|^2$, and δ is a small positive real number. As a matter of notation, we use v^T to denote the gradient

$$v^T \equiv \nabla f(x). \tag{2}$$

It can be seen from (1) that the norm of v is an appropriate measure of the local sensitivity of f. For simplicity of exposition, we assume throughout this paper that all functions considered are at least twice continuously differentiable.

Somewhat surprisingly, rigorous probability arguments show that only a few function evaluations are needed to obtain accurate and reliable estimates of the norm of v. In this paper the small-sample method of [24] is applied to the problem of condition estimation for more general matrix functions such as the matrix exponential map $X \mapsto e^X$ and the map $(A, F, G) \mapsto X$, where X satisfies an algebraic Riccati equation

$$0 = G + A^T X + XA - XFX. \tag{3}$$

This is illustrated in Section 2 with many other examples. That section also compares the small-sample statistical method with the more traditional power method of condition estimation. Short MATLAB routines are given in Section 2 for estimating the sensitivity of general matrix functions. The theoretical basis for the small-sample methods is briefly described in the remainder of this section and is given in more detail in [24].

The central idea of this method is that the norm of the gradient can be estimated if we can afford another function evaluation beyond $f(x)$. Suppose, for example, that we evaluate f at $x + \delta z$. If z has unit norm, then the Newton difference defined by

$$dz \equiv \frac{f(x + \delta z) - f(x)}{\delta} \tag{4}$$

satisfies

$$dz = v^T z + O(\delta). \tag{5}$$

If z is selected uniformly and randomly from the unit sphere S_{n-1} in n dimensions, then (see Theorem 1 in [24]) the expected value of $|v^T z|$ is given by

$$E(|v^T z|) = \|v\| E_n, \tag{6}$$

where $E_1 = 1$, and for $n > 1$

$$E_n = \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \quad \text{for } n \text{ odd} \tag{7}$$

$$E_n = \frac{2}{\pi} \frac{2 \cdot 4 \cdot 6 \cdots (n-2)}{1 \cdot 3 \cdot 5 \cdots (n-1)} \quad \text{for } n \text{ even.} \tag{8}$$

For large n it is convenient to use the approximation $E_n \approx \sqrt{2/(\pi(n - 1/2))}$, which is accurate to three significant digits for $n \geq 10$.

The condition estimator $|dz|/E_n$ has expected value equal to the true condition number $\|v\|$ plus a term of order δ. Typically, we can take δ sufficiently small (say, less than 10^{-8}), so that for the purposes of estimating $\|v\|$ the expected value of $|dz|/E_n$ is more than adequate. To avoid having to add terms of order δ to all of our results, we will now make the simplifying assumption that for any given vector $z \in S_{n-1}$ we are able to evaluate $v^T z$ with the understanding that in real life we are merely able to evaluate dz as in (4). This is reasonable when we remember that for most sensitivity estimates, we only need to know the true condition number to within a factor of 10 or so, and we can often tolerate errors in the estimate up to a factor of 100.

What is the probability that the estimator

$$\zeta \equiv |v^T z|/E_n$$

lies within a given factor w of $\|v\|$? This question was considered in detail in [24] and it was found that because ζ^2 has a Beta distribution,

$$\Pr(\|v\|/w \leq \zeta \leq w\|v\|) \geq 1 - \frac{2}{\pi w} + O(\frac{1}{w^2}). \tag{9}$$

Thus ζ is a linear or first-order condition estimate in the sense that the chance of a catastrophically low or high estimate is inversely proportional to the size of the error. For example, $\Pr(\|v\|/100 \leq \zeta \leq 100\|v\|) \geq 0.9936$, so that the chance of being off by more than a factor of 100 is less than 1 in 100.

While this is good, there are some situations in which we need more reliability. One way to achieve this is to use more function evaluations to get different values $\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(m)}$ corresponding to independently randomly generated vectors $z^{(1)}, z^{(2)}, \ldots, z^{(m)}$ in S_{n-1} and then to take the average

$$\zeta(m) \equiv \frac{\zeta^{(1)} + \cdots + \zeta^{(m)}}{m}. \tag{10}$$

For $w > 1$, we have from [24] that

$$\Pr(\|v\|/w \leq \zeta(m) \leq w\|v\|) \geq 1 - \frac{1}{m!}\left(\frac{2m}{\pi w}\right)^m + O(\frac{1}{w^{m+1}}). \tag{11}$$

with asymptotic equality as $w \to +\infty$ or $m \to +\infty$. Thus $\zeta(m)$ is an mth-order condition estimator. That is, the probability that the estimator is off by more than a factor of w is less than a constant times w^{-m}. For example, with $m = 3$, which corresponds to three extra function evaluations, we have $\Pr(\|v\|/10 \leq \zeta(3) \leq 10\|v\|) \geq 0.99884$, so that the chance of being off by a factor of 10 or more is about 1 in 862. Similarly, the chance of being off by a factor of 100 or more is approximately one in a million. Hence very reliable condition estimates can be obtained with only a few extra function evaluations.

Simple Algorithms for Affine Invariant Image Processing

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Abstract

Exact solutions to a PDE associated with affine invariant image processing are derived for initial conditions corresponding to extrema and saddle points. For these initial conditions the original nonlinear PDE reduces to the standard convection equation. Application of the affine invariance property then generates a much larger family of exact solutions, including those with elliptically symmetric initial conditions. Examination of this family clarifies surface evolution near extrema and saddle points in terms of flows along characteristics: surface information flows toward extrema and away from saddle points. This viewpoint also leads to a nonlinear Courant number relating the permissable evolution step size to a given spatial step size. The exact solution for circularly symmetric initial conditions shows that standard central difference approximations of the affine invariant PDE can produce incorrect results in the form of spikes at extrema. A procedure for avoiding this problem is discussed and short MATLAB routines for affine image processing are given.

A Simple Method of Generating a Basis of Solutions for Linear Partial Differential Equations

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Abstract

A simple procedure is given for constructing a complete basis of solutions to a given linear constant coefficient partial differential equation. A variety of examples are presented including eigenvalues problems such as the Helmholtz equation.

Numerical Methods for Studying Parameter Dependence of Solutions to Schrödinger's Equation

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Abstract

Numerical methods are presented for handling two of the most important computational problems associated with the study of parameter dependence of solutions to the periodically time dependent Schrödinger equation as approximated by a finite system of ordinary differential equations (ODEs): 1) lengthy integration times due to the growth in parameters as the number of states increase and 2) repeated integrations over a grid in parameter space. These methods are designed to work together and are independent of the numerical algorithm used to integrate the ODE system. The first method uses a simple diagonal transformation of the solution to control the magnitude of terms appearing in the differential system, and the second method relies on matrix interpolation together with a Taylor's series expansion to reduce the number of integrations in parameter space. The combination of these two procedures results in a dramatic decrease in computation time for an example problem of an electron confined to a quantum well.

Keywords: Schrödinger's Equation, Parameter Study, Matrix Interpolation
AMS(MOS) subject classification: 65L05, 65F30.
Abbreviated title: Integrating Schrödinger's Equation

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METHODS FOR THE NUMERICAL INTEGRATION OF HAMILTONIAN SYSTEMS*

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Abstract. To compute an infinite horizon optimal controller for a linear periodic system via an invariant subspace method, the computation of the period map associated with the Hamilton-Jacobi-Bellman equations is required. In this paper we discuss methods for the numerical integration of such Hamiltonian systems. Two numerical integration techniques are introduced. A method is developed whereby symplectic invariants associated with the Hamilton-Jacobi-Bellman equations are preserved. Also, a shifting scheme is introduced that in effect swaps the roles of the stable and unstable invariant subspaces by using the semigroup property of state transition matrices. A shift is introduced into the resultant initial value problem ensuring that the eigenvalues of a differential equation reside in the region of absolute stability for an appropriate numerical integration routine. These techniques are then compared to standard numerical integration routines to ascertain their efficiency and accuracy.

1. Introduction

In this paper, we examine various methods of computing the optimal feedback controller gains for linear, periodic time-varying systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

with quadratic cost functional:

$$\min_{u} \frac{1}{2} \int_0^\infty \left[x^T Q(t)x + u^T R(t)u \right] dt \tag{2}$$

Trace Norm Bounds for Stable Lyapunov Operators *

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Abstract

Certain aspects of stable Lyapunov operators can be easily studied by exploiting the linearity of the trace operator and its invariance under reversal of order in matrix products. For example, sharp upper and lower bounds on the trace of solutions to the stable Lyapunov equation can be obtained by applying the trace operator to a well known integral representation of these solutions.

Other applications include using the connection between dual norms and the trace operator to obtain new results on the norms of Lyapunov operators associated with the conditioning of solutions to the Riccati equation. In this regard, trace norm results can be obtained from well known spectral norm results since the trace and spectral norms are dual to each other. A somewhat deeper analysis involving the power method gives monotonically decreasing upper bounds on the Frobenius norms of these Lyapunov operators; these upper bounds complement the usual monotonically increasing lower bounds associated with the power method and provide a nice means of assessing the accuracy of the resulting Frobenius norm estimates.

Keywords: Trace, Lyapunov Equation, Dual Norms.
AMS(MOS) subject classification: 65F35.
Abbreviated title: Lyapunov Trace Bounds

A hyperbolic tangent identity and the geometry of Padé sign function iterations*

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The rational iterations obtained from certain Padé approximations associated with computing the matrix sign function are shown to be equivalent to iterations involving the hyperbolic tangent and its inverse. Using this equivalent formulation many results about these Padé iterations, such as global convergence, the semi-group property under composition, and explicit partial fraction decompositions can be obtained easily. In the second part of the paper it is shown that the behavior of points under the Padé iterations can be expressed, via the Cayley transform, as the combined result of a completely regular iteration and a chaotic iteration. These two iterations are decoupled, with the chaotic iteration taking the form of a multiplicative linear congruential random number generator where the multiplier is equal to the order of the Padé approximation.

1. Introduction

A new family of rational iterations was introduced in [10] for computing the matrix sign function. These iterations are based on Padé approximations for a certain hypergeometric function. Table 1 of [10] gives the first few such formulas and it is proved there that the formulas whose numerator and denominator polynomials differ in degree by 1 are globally convergent. In the sequel we shall refer to the iteration formulas defined by such rational functions as the *principal Padé iterations*. For convenience, the theory described in this paper will be carried out for the scalar case. However, it must be emphasized that the scalar theory carries over directly to the matrix case (see [10] for details).

In this paper two interesting aspects of the principal Padé iterations are considered. The first is an expression of these iterations in terms of the hyperbolic tangent and its inverse, while the second deals with chaotic and regular behavior of points

The Matrix Sign Function *

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Abstract

A survey of the matrix sign function is presented, including the historical background, definitions and properties, approximation theory and computational methods, condition theory and estimation procedures, as well as application to the areas of control theory, eigendecompositions and roots of matrices. Many new results are presented along with a description of the Green's matrix theory developed by Godunov and others.

1 Introduction

The matrix sign function can be used to find bases for the positive and negative invariant subspaces of a matrix. Since many important problems in the areas of control theory and eigenvalue-eigenvector decompositions have solutions that can be given in terms of these subspaces, the matrix sign function has enjoyed renewed interest in the last twenty years. This paper summarizes the developments of this period and then looks at the theory and applications of the sign function.

In 1877 Zolotarjov [98] characterized the best rational approximations of the scalar sign function in terms of elliptic sine functions. However, this result does not seem to be well known and the introduction of the matrix sign function, especially into the area of control theory,

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Scaling Function Identification and Approximation *

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Abstract

An algorithm is presented for recovering the scaling coefficients from a given scaling function. This procedure is then extended to the problem of finding scaling functions that approximate a given function.

1 Introduction

Given $c_0, c_1, \ldots, c_n$ such that

$$\sum_{k=0}^{n} c_k = 2 , \tag{1}$$

define the associated scaling function ϕ by first setting $\phi(x) = 0$ for x outside the interval $(0, n)$. For $x \in (0, n)$ set

$$\phi(x) = \sum_{k=0}^{n} c_k \phi(2x - k) , \tag{2}$$

subject to the normalization condition

$$\int_{-\infty}^{\infty} \phi(x) dx = 1 . \tag{3}$$

Necessary and sufficient conditions for ϕ to be in $L^1[0, n]$ are given by Daubechies and Lagarias [1] and a well known dyadic procedure for constructing $\phi(x)$ at rational points $x \in (0, n)$ from the values $c_0, c_1, \ldots, c_n$ is given by Strang in [2]. See also [3].

In this note we consider the inverse problem: if we are given a scaling function ϕ, how do we recover the values of the scaling coefficients $c_0, c_1, \ldots, c_n$? The answer to this question opens up the possibility of finding scaling functions which approximate more general functions.

The key observation is that c_0 can be recovered by using (2) for x in the interval $(0, 1/2)$. Since $\phi(x) = 0$ for $x \leq 0$, if $x \in (0, 1/2)$ then

$$\phi(x) = c_0 \phi(2x) . \tag{4}$$